

Gauge Theory Webs and Surfaces

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We analyze the perturbative cusp for massless gauge theories in coordinate space, and express it as the exponential of a two-dimensional integral. The exponent has a geometric interpretation, which links the renormalization scale with invariant distances.

Introduction. Wilson lines, or ordered exponentials [1, 2] represent the interaction of energetic partons with relatively softer radiation in gauge theories. For constant velocities, ordered exponentials of semi-infinite length correspond to the eikonal approximation for energetic partons. Classic phenomenological applications of ordered exponentials include soft radiation limits in electron-positron annihilation and deeply inelastic scattering [3]. In these cases, the electroweak current is represented by a color singlet vertex at which lines with different velocities are coupled. This vertex is often referred to as a cusp. The set of all virtual corrections for the cusp is formally identical to a vacuum expectation value, and can be written as

$$\Gamma^{(f)}(\beta_1, \beta_2) = \left\langle T \left(\Phi_{\beta_2}^{(f)}(\infty, 0) \Phi_{\beta_1}^{(f)}(0, -\infty) \right) \right\rangle_0, \quad (1)$$

in terms of constant-velocity ordered exponentials,

$$\begin{aligned} \Phi_{\beta_i}^{(f)}(x + \lambda \beta_i, x) \\ = \mathcal{P} \exp \left(-ig \int_0^\lambda d\lambda' \beta_i \cdot A^{(f)}(x + \lambda' \beta_i) \right). \end{aligned} \quad (2)$$

Here f labels a representation of the gauge group (which we suppress below) and β_i is a four-velocity, taken light-like in the following. We will explore all-orders properties of single- and multiple-cusp products of ordered exponentials, computed perturbatively in coordinate space.

Perturbative corrections to the cusp, Eq. (1) are scaleless, and hence vanish in dimensional regularization. The ultraviolet poles of (1) determine the anomalous dimension of the cusp, but for an asymptotically free theory neither its ultraviolet nor its infrared behavior can be considered as truly physical. At very short distances, dynamics is perturbative and recoil cannot be neglected. At very long distances, it is nonperturbative and dominated by the hadronic spectrum. In this discussion, we will regard the cusp as an interpolation between these asymptotic regimes, and we will concentrate on the structure of the integrals in the intermediate region.

We will argue that in any gauge theory the cusp matrix element can be expressed as the exponential of an integral over a two-dimensional surface. The corresponding integrand is an expansion in the gauge theory coupling, evaluated at a scale given by the invariant distance from a point on the surface to the cusp vertex. We will go on to apply this result to multi-cusp polygonal Wilson loops [4, 5].

Exponentiation and Webs. The cusp has long been known [6] to be the exponential of a sum of special diagrams called webs, which are irreducible by cutting two eikonal lines. We represent this result as

$$\Gamma(\beta_1, \beta_2, \varepsilon) = \exp E(\beta_1, \beta_2, \varepsilon), \quad (3)$$

in $D = 4 - 2\varepsilon$ dimensions. The exponent E equals a sum over web diagrams, d , each given by a group factor multiplied by a diagrammatic integral,

$$E(\beta_1, \beta_2, \varepsilon) = \sum_{\text{webs } d} \bar{C}_d \mathcal{F}_d(\beta_1, \beta_2, \varepsilon), \quad (4)$$

where \mathcal{F}_d represents the momentum- or coordinate-space integral for diagram d . The coefficients of these integrals, \bar{C}_d are modified color factors. Two-loop examples are shown in Fig. 1. In momentum space we can write the exponent E as an integral over a single, overall loop momentum integral that connects the web with the cusp vertex, assuming that all loop momentum integrals internal to the web have been carried out, including as well the necessary counterterms of the theory [7, 8]. Taking into account the boost invariance of the cusp, and the invariance of the ordered exponentials under rescalings of the velocities β_i , we have for the exponent the form,

$$E = \int \frac{d^D k}{(2\pi)^D} \bar{w} \left(k^2, \frac{k \cdot \beta_1 k \cdot \beta_2}{\beta_1 \cdot \beta_2}, \mu^2, \alpha_s(\mu^2, \varepsilon), \varepsilon \right). \quad (5)$$

Multiplicative renormalization of the cusp implies additive renormalization for the exponent E . In addition, the webs themselves are renormalization-scale independent, $\mu \frac{d}{d\mu} \bar{w} \left(k^2, \frac{k \cdot \beta_1 k \cdot \beta_2}{\beta_1 \cdot \beta_2}, \mu^2, \alpha_s(\mu^2, \varepsilon), \varepsilon \right) = 0$. A further property of webs is the absence of collinear and soft subdivergences. That is, in Eq. (5), collinear poles are generated only when k^2 and either $k \cdot \beta_1$ or $k \cdot \beta_2$ vanish, infrared poles only when all three vanish and the

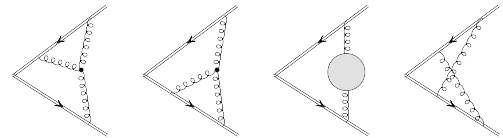


FIG. 1: Two-loop web diagrams, all with the same modified color factor $\bar{C}_d = C_A C_A$. Notice that only for the rightmost diagram does the modified color factor differ from its original color factor.

overall ultraviolet poles only when all components of k diverge. Eq. (5) thus organizes the same double poles found in the corresponding partonic form factors [8, 9]. A proof of these properties in momentum space is given in Ref. [8], based on the factorization of soft gluons from fast-moving collinear partons. These considerations suggest that when embedded in an on-shell amplitude, the web acts as a unit, almost like a single gluon, dressed by arbitrary orders in the coupling. In the following, we observe that this analogy can be extended to coordinate space.

The coordinate space analog of Eq. (5) is a double integral over two parameters λ and σ that measure distances along the Wilson lines β_1 and β_2 , respectively, with a new web function, w , which depends on these variables through the only available dimensionless combination $\beta_1 \cdot \beta_2 \lambda \sigma \mu^2$,

$$E = \int_0^\infty \frac{d\lambda}{\lambda} \int_0^\infty \frac{d\sigma}{\sigma} w(\alpha_s(\mu^2, \varepsilon), \lambda \sigma \mu^2, \varepsilon), \quad (6)$$

where here and below, we set $\beta_1 \cdot \beta_2 = 1$ and choose time-like kinematics. We emphasize that we are interested primarily in the form and symmetries of the integrand, rather than its convergence properties. Nevertheless, to separate infrared and ultraviolet poles in the integration, it is necessary that the integrand, w in Eq. (6) be free of both infrared and ultraviolet divergences at $\varepsilon = 0$ in renormalized perturbation theory (aside from the renormalization of the cusp itself). For finite values of λ and σ , there are no infrared divergences from integrations over the internal vertices of webs [10]. We will see below that for nonzero λ and σ , w also has no collinear subdivergences. All ε poles of the exponent, and therefore the cusp, are then associated with the integrals over λ and σ in (6).

To derive Eq. (6) along with the finiteness of w , we follow Ref. [11] and write the exponent as a sum over the numbers, e_a , of gluons attached to the two Wilson lines, of velocity β_a , $a = 1, 2$. For a given choice of e_a , the web diagrams are integrals over the positions $\tau_{j_a}^{(a)}$ of these ordered vertices of a function $\mathcal{W}_{e_1, e_2}(\tau_{j_a}^{(a)})$ found by integrating over all the internal vertices of the corresponding web diagrams. In the notation of Ref. [11] we then have at i th order ($i \geq e_1 + e_2$),

$$E^{(i)} = \sum_{e_1, e_2=1}^{i-1} \prod_{a=1}^2 \prod_{j_a=1}^{e_a} \int_{\tau_{j_a-1}^{(a)}}^{\infty} d\tau_{j_a}^{(a)} \mathcal{W}_{e_1, e_2}^{(i)}(\{\tau_{j_a}^{(a)}\}), \quad (7)$$

where $\tau_0^{(a)} \equiv 0$. Here and below we expand functions as $E = \sum (\alpha_s/\pi)^i E^{(i)}$. Next, we fix the values of the vertices in Eq. (7) that are furthest from the origin, and denote them as $\tau_{e_1}^{(1)} = \lambda$, $\tau_{e_2}^{(2)} = \sigma$. The function w in

Eq. (6) is then defined by

$$w^{(i)}(\lambda \sigma) = \sum_{e_1, e_2} \prod_{j_1=1}^{e_1-1} \int_{\tau_{j_1-1}^{(1)}}^{\lambda} d\tau_{j_1}^{(1)} \prod_{j_2=1}^{e_2-1} \int_{\tau_{j_2-1}^{(2)}}^{\sigma} d\tau_{j_2}^{(2)} \times \mathcal{W}_{e_1, e_2}^{(i)}(\{\tau_{j_a}^{(a)}\}), \quad (8)$$

where we integrate over all $\tau_{j_a}^{(a)}$, $j_a < e_a$. We want to show that these integrals, along with the integrals over internal vertices of the web diagrams, do not give rise to singular behavior in the function $w(\sigma \lambda)$, once the full set of web diagrams is combined at a given order.

Diagram by diagram, one may use an analysis of the analytic structure of the coordinate integrations [12] combined with a coordinate space power counting technique to identify the most general singular subregions in coordinate space [10]. These occur whenever a subset of internal vertices of a web diagram becomes “collinear” by approaching either Wilson line, while other, “soft” vertices remain at finite distances.

In coordinate space, as in momentum space, however, soft and collinear vertices are always attached to each other through lines that carry unphysical polarizations. Region by region, soft and collinear subdiagrams then factorize [13, 14]. Once in factorized form, divergent integrals cancel when all web diagrams are combined at a given order [10] by a method related to the momentum space arguments given in [8]. It is necessary to implement this cancellation at fixed λ and σ . Once this is done and subdivergences thereby eliminated, the integrals over all vertices of the web diagrams converge on scales set by σ and λ in (6), and the web acts as a unit. Singular behavior of the cusp arises as σ and λ vanish, and in these limits all web vertices approach the directions of β_1 or β_2 together, as in Fig. 2. This is the perturbative realization of the web as a geometrical object.

The web function w constructed this way is scale invariant, so that in (6), we may shift the renormalization scale to the product $(\sigma \lambda)^{-1}$, which results in an expression with the coupling running as the leading vertices move up and down the Wilson lines,

$$E = \int_0^\infty \frac{d\lambda}{\lambda} \int_0^\infty \frac{d\sigma}{\sigma} w(\alpha_s(1/\lambda \sigma, \varepsilon), \varepsilon). \quad (9)$$

In this all-orders form, dependence on the product $\lambda \sigma$ is entirely through the running coupling, aside from the overall dimensional factor. For conformal field theories,



FIG. 2: Representation of singular regions for a two-loop web diagram.

Eq. (9) for the cusp holds as well at strong coupling [15–17], where the coordinates λ and σ also parameterize a surface. The generality of these results can be traced to the symmetries of the problem [17]. It is interesting to note, however, that in the strong coupling analysis, the product of internal coordinates $\lambda\sigma$, which serves as the renormalization scale in Eq. (9), relates the plane of the Wilson lines to a minimal surface in five dimensions.

The lowest order expression for Eq. (6) is found directly from the coordinate space gluon propagator in Feynman gauge, $D^{\mu\nu} = \frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}} \frac{-g^{\mu\nu}}{(-x^2+i\epsilon)^{1-\varepsilon}}$. The resulting expression already illustrates the nontrivial relationship between the renormalization scale and the positions of the vertices,

$$E^{(\text{LO})} = -C_a \Gamma(1-\varepsilon) \frac{\alpha_s(\mu)}{2\pi} \int_0^\infty \frac{d\lambda}{\lambda} \frac{d\sigma}{\sigma} (2\pi\lambda\sigma\mu^2)^\varepsilon, \quad (10)$$

with $C_a = 4/3, 3$ for $a = q, g$ in QCD. More generally, consistency with momentum space pole structure [3, 8] requires

$$w(\alpha_s(1/[\lambda\sigma]), \varepsilon) = -\frac{1}{2} \Gamma_{\text{cusp}}(\alpha_s(1/[\lambda\sigma], \varepsilon)) + \mathcal{O}(\varepsilon), \quad (11)$$

where $\Gamma_{\text{cusp}}(\alpha_s)$ is the cusp anomalous dimension, defined for flavor a as in [4, 18], $\Gamma_{\text{cusp}}^{(a)} = (\alpha_s/\pi)C_a[1 + (\alpha_s/\pi)K]$, plus higher orders, with $K = [C_A(67/36 - \pi^2/12) - (5n_f)/18]$. We have verified this relation for the coordinate web constructed as above at two loops [10]. Interpreted as a fully scaleless integral, Eq. (11) can be defined by its ultraviolet poles, and is of course gauge invariant. If the integrals are cut off in the infrared, however, gauge-dependent corrections at $\mathcal{O}(\varepsilon)$ in Eq. (11) remain in general.

Polygon Loops. The above reasoning leads to a number of interesting results for polygonal closed Wilson loops [5, 15, 16]. These amplitudes also exponentiate in perturbation theory in terms of webs [5]. To this observation we may apply once again the lack of subdivergences for webs.

Generic diagrams for polygonal loops are shown in Figs. 3 and 4. In Fig. 3, for example, the a th vertex of the polygon represents a cusp vertex that connects two Wilson lines, of velocity β_a and β'_a .

Exponentiation in coordinate space implies that the logarithm of the polygon P is a sum of the web configu-

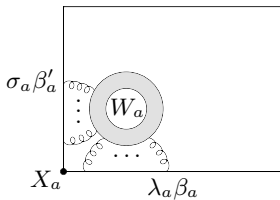


FIG. 3: A single-cusp web W_a , in the sum of Eq. (12).

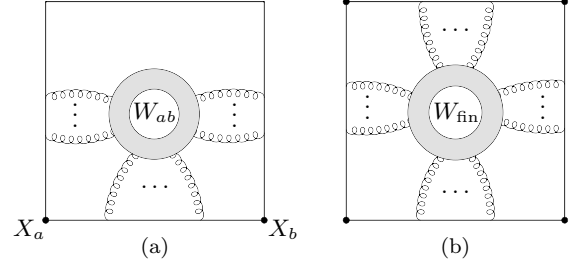


FIG. 4: (a) A ‘side’ web W_{ab} in of Eq. (12), in this case associated with the light-like side $X_a - X_b$. (b) A web that contributes to W_{plane} in Eq. (12).

rations represented by the figures,

$$\ln P = \sum_{\text{cusps } a} W_a + \sum_{\text{sides } ab} W_{ab} + W_{\text{plane}}. \quad (12)$$

The first terms organize webs associated entirely with one of the cusps of the polygon, constructed in terms of the coordinate webs identified above. Because each edge is of finite length, there are now additional terms associated with the end-point contributions, which may be combined with webs connecting three sides. The cancellation of subdivergences in webs implies that after a sum over diagrams, only the cusp poles and a single, overall collinear singularity survives [5, 10]. There remains a finite contribution from webs that connect all four (or in general more) of the Wilson lines, and these are represented by the final term in (12).

Evidently, $W_a(\beta_a, \beta'_a)$ is the same as for the finite Wilson lines in Eq. (9), in terms of the lengths of the sides of the polygon, which can be scaled to some constant, X ,

$$W_a(\beta_a, \beta'_a, X) = - \int_0^X \frac{d\lambda_a}{\lambda_a} \int_{-X}^0 \frac{d\sigma_a}{\sigma_a} w(\alpha_s(1/\lambda_a\sigma_a, \varepsilon), \varepsilon). \quad (13)$$

The web function can depend only on the scalar products of the velocities, and we may assume for simplicity that these are all the same.

Polygons of this sort have been studied in the context of a duality to scattering amplitudes [5, 15]. Here, we consider a four-sided polygon that projects to a square in the x_1/x_2 plane, with side X , as in Figs. 3–4. In four dimensions, the loop starts at the origin, travels along the plus- x^1 direction for a ‘time’ $X^0 = X$, then changes direction to x^2 for time X , and then moves backwards in time and space, first in the x^1 direction, then x^2 , back to the origin. We can now use the coordinates x^1 and x^2 to define parameters λ_a and σ_a for each of the cusp integrals W_a in Eq. (13),

$$\begin{aligned} \sigma_1 &= -x_2, & \lambda_1 &= x_1, \\ \sigma_2 &= x_1 - X, & \lambda_2 &= x_2, \\ \sigma_3 &= x_2 - X, & \lambda_3 &= X - x_1, \\ \sigma_4 &= -x_1, & \lambda_4 &= X - x_2. \end{aligned} \quad (14)$$

In this notation, we can add the four cusp web integrals of Eq. (13), to get a single integral over x_1 and x_2 . The web functions, of course, depend on the particular forms of λ and σ above. We find

$$\sum_{a=1}^4 W_a(\beta_a, \beta'_a) = \int_0^X dx_1 \int_0^X dx_2 \frac{(X-x_2)[(X-x_1)w_1 + x_1w_2] + x_2[x_1w_3 + (X-x_1)w_4]}{x_1(X-x_1)x_2(X-x_2)}, \quad (15)$$

where $w_a \equiv w(\alpha_s(\lambda_a(x_1, x_2)\sigma_a(x_1, x_2)))$. For a conformal theory, all dependence on the σ_a and λ_a is in the denominators and we can sum over a to get a result in terms of a constant web function w_0 . Changing variables to $y_a = 1 - 2x_a/X$, we derive the unregularized form found from the analysis of extremal two-dimensional surfaces embedded in a five-dimensional background in [15],

$$\sum_{a=1}^4 W_a(\beta_a, \beta'_a) = \int_{-1}^1 dy_1 \int_{-1}^1 dy_2 \frac{4w_0}{(1-y_1^2)(1-y_2^2)}, \quad (16)$$

to which we should add the collinear and finite multi-cusp contributions of Fig. 4.

Conclusions. We have found that when the massless cusp is analyzed in coordinate space, it is naturally written as the exponential of a two-dimensional integral. The integrand, a web function, depends on the single invariant scale through the running of the coupling, which for a theory that is conformal in four dimensions agrees with strong coupling results [15–17]. This agreement extends to aspects of closed, polygonal Wilson loops. These results do not rely on a planar limit [19], but it is natural to conjecture that for large N_c the integral may take on an even more direct interpretation in terms of surfaces for non-conformal theories.

In QCD, of course, our explicit knowledge of the web function is limited to the first few terms in the perturbative series, which run out of predictive power as the invariant distance increases. The functional form, however, holds to all orders in perturbation theory, and may point to an interpolation between short and long distances.

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